

# The Similarity Invariants of non-lightlike curves in the Minkowski 3-space

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## Abstract

In this paper, we firstly introduce the group of similarity transformations in the Minkowski-3 space. We describe differential-geometric invariants of a non-lightlike curve according to the group of similarity transformations of the Minkowski 3-space. We show extension of fundamental theorem for non-lightlike curves under the group of similarity of the Minkowski 3-space.

**Keywords :** Minkowski space, Similarity invariants, non-lightlike curves.  
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## 1 Introduction

A similarity transformation (or similitude) of Euclidean space, which consists of a rotation, a translation and an isotropic scaling, is an automorphism preserving the angles and ratios between lengths. The geometric properties unchanged by similarity transformations is called the *similarity geometry*. The whole Euclidean geometry can be considered as a glass of similarity geometry. The similarity transformations are studying in most area of the pure and applied mathematics.

Curve matching is an important research area in the computer vision and pattern recognition, which can help us determine what category the given test curve belongs to. Also, the recognition and pose determination of 3D objects can be represented by space curves is important for industry automation, robotics, navigation and medical applications. S. Li [21] showed an invariant representation based on so-called similarity-invariant coordinate system (SICS) for matching 3D space curves under the group of similarity transformations. He also [22] presented a system for matching and pose estimation of 3D space curves under the similarity transformation. Brook et al. [1] discussed various problems of image processing and analysis by using the similarity transformation. Sahbi [6] investigated a method for shape description based on kernel principal component analysis (KPCA) in the similarity invariance of KPCA. There are many applications of the similarity transformation in the computer vision and pattern recognition (see also [5, 8]).

The idea of self-similarity is one of the most basic and fruitful ideas in mathematics. A self-similar object is exactly similar to a part of itself, which in turn remains similar to a smaller part of itself, and so on. In the last few decades it established itself as the central notion in areas such as fractal geometry, dynamical systems, computer networks and statistical physics. Mandelbrot presented the first description of self-similar sets, namely sets that may be expressed as unions of rescaled copies of themselves. He called these sets fractals, which are systems that present such self-similar behavior and the examples in nature are many. The Cantor set, the von Koch snowflake curve and the Sierpinski gasket are some of the most famous examples of such sets. Hutchinson and, shortly thereafter, Barnsley and Demko showed how systems of contractive maps with associated probabilities, referred to as Iterated Function Systems (IFS), can be used to construct fractal, self-similar sets and measures supported on such sets (see [2, 7, 12, 13, 14]).

When Euclidean 3-space is endowed with Lorentzian inner product, we obtain *Lorentzian similarity geometry*. Lorentzian flat geometry is inside the Lorentzian similarity geometry. Kamishima [24] studied the properties of compact Lorentzian similarity manifolds using developing maps and holonomy representations. The geometric invariants of curves in the Lorentzian similarity geometry have not been considered so far. The theme of similarity and self-similarity will be interesting in the Lorentzian-Minkowski space.

Many integrable equations, like Korteweg-de Vries (mKdV), sine-Gordon and nonlinear Schrödinger (NLS) equations, in soliton theory have been shown to be related to motions of inextensible curves in the Euclidean space. By using the similarity invariants of curves under the similarity motion, KS. Chou and C. Qu [11] showed that the motions of curves in two-, three- and  $n$ -dimensional ( $n > 3$ ) similarity geometries correspond to the Burgers hierarchy, Burgers-mKdV hierarchy and a multi-component generalization of these hierarchies in  $\mathbb{E}^n$ . Moreover, to study the motion of curves in the Minkowski space also attracted researchers' interest. Gürses [16] studied the motion of curves on two-dimensional surface in Minkowski 3-space. Q. Ding and J. Inoguchi [19] showed that binormal motions of curves in Minkowski 3-space are equivalent to some integrable equations. Therefore, the current paper will contribute to study the motion of curves with similarity invariants in  $\mathbb{E}_1^3$ .

The broad content of similarity transformations were given by [15] in arbitrary-dimensional Euclidean spaces. Differential geometric invariants of Frenet curves up to the group of similarities were studied by [20] in the Euclidean 3-space. In current paper, Lorentzian version of similarity transformations will be entitled by pseudo-similarity transformation defined by (1) in the section 2. The main idea of this paper is to extend the fundamental theorem for a non-null curve with respect to p-similarity motion and determine non-null self-similar curves in the Minkowski 3-space.

The content of paper is as follows. We prove that p-similarity transformations preserve the causal characters of vectors and the angles in  $\mathbb{E}_1^3$ . We examine invariants of a non-lightlike Frenet curve up to the group of p-similarities. We also show the relationship between the focal curvatures and these invariants for non-lightlike Frenet curves in  $\mathbb{E}_1^3$ . We give the uniqueness theorem which states that two non-lightlike Frenet curves having same the p-shape curvature and same the p-shape torsion are equivalent modulo a p-similarity. Furthermore, we obtain the existence theorem that is a procedure for construction of a non-lightlike Frenet curve by means of its p-shape curvature and p-shape torsion under some initial conditions. Lastly, we give examples about construction of a non-lightlike Frenet curve with a given p-shape.

## 2 The Fundamental Group of Lorentzian Similarity Geometry

Firstly, let us give some basic notions of the Lorentzian geometry. Let  $\mathbf{x} = (x_1, x_2, x_3)^T$ ,  $\mathbf{y} = (y_1, y_2, y_3)^T$  and  $\mathbf{z} = (z_1, z_2, z_3)^T$  be three arbitrary vectors in the Minkowski space  $\mathbb{E}_1^3$ . The Lorentzian inner product of  $\mathbf{x}$  and  $\mathbf{y}$  can be stated as  $\mathbf{x} \cdot \mathbf{y} = \mathbf{x}^T I^* \mathbf{y}$  where  $I^* = \text{diag}(-1, 1, 1)$ . The vector  $\mathbf{x}$  in  $\mathbb{E}_1^3$  is called a spacelike vector, lightlike (or null) vector and timelike vector if  $\mathbf{x} \cdot \mathbf{x} > 0$  or  $\mathbf{x} = 0$ ,  $\mathbf{x} \cdot \mathbf{x} = 0$  or  $\mathbf{x} \cdot \mathbf{x} < 0$ , respectively. The norm of the vector  $\mathbf{x}$  is described by  $\|\mathbf{x}\| = \sqrt{|\mathbf{x} \cdot \mathbf{x}|}$ . The Lorentzian vector product  $\mathbf{x} \times \mathbf{y}$  of  $\mathbf{x}$  and  $\mathbf{y}$  is defined as follows:

$$\mathbf{x} \times \mathbf{y} = \begin{bmatrix} -i & j & k \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}$$

The hyperbolic and Lorentzian unit spheres are

$$H_0^2 = \{\mathbf{x} \in \mathbb{E}_1^3 : \mathbf{x} \cdot \mathbf{x} = -1\} \text{ and } S_1^2 = \{\mathbf{x} \in \mathbb{E}_1^3 : \mathbf{x} \cdot \mathbf{x} = 1\}$$

respectively. There are two components  $H_0^2$  passing through  $(1, 0, 0)$  and  $(-1, 0, 0)$  a future pointing hyperbolic unit sphere a past pointing hyperbolic unit sphere, and they are denoted by  $H_0^{2+}$  and  $H_0^{2-}$ , respectively (see [3] and [23]).

**Theorem 1** Let  $\mathbf{x}$  and  $\mathbf{y}$  be vectors in the Minkowski 3-space.

- (i) If  $\mathbf{x}$  and  $\mathbf{y}$  are future-pointing (or past-pointing) timelike vectors, then  $\mathbf{x} \times \mathbf{y}$  is a spacelike vector,  $\mathbf{x} \cdot \mathbf{y} = -\|\mathbf{x}\| \|\mathbf{y}\| \cosh \theta$  and  $\mathbf{x} \times \mathbf{y} = \|\mathbf{x}\| \|\mathbf{y}\| \sinh \theta$  where  $\theta$  is the hyperbolic angle between  $\mathbf{x}$  and  $\mathbf{y}$ .
- (ii) If  $\mathbf{x}$  and  $\mathbf{y}$  are spacelike vectors satisfying the inequality  $|\mathbf{x} \cdot \mathbf{y}| < \|\mathbf{x}\| \|\mathbf{y}\|$ , then  $\mathbf{x} \times \mathbf{y}$  is timelike,  $\mathbf{x} \cdot \mathbf{y} = \|\mathbf{x}\| \|\mathbf{y}\| \cos \theta$  and  $\mathbf{x} \times \mathbf{y} = \|\mathbf{x}\| \|\mathbf{y}\| \sin \theta$  where  $\theta$  is the angle between  $\mathbf{x}$  and  $\mathbf{y}$ .
- (iii) If  $\mathbf{x}$  and  $\mathbf{y}$  are spacelike vectors satisfying the inequality  $|\mathbf{x} \cdot \mathbf{y}| > \|\mathbf{x}\| \|\mathbf{y}\|$ , then  $\mathbf{x} \times \mathbf{y}$  is spacelike,  $\mathbf{x} \cdot \mathbf{y} = \|\mathbf{x}\| \|\mathbf{y}\| \cosh \theta$  and  $\mathbf{x} \times \mathbf{y} = \|\mathbf{x}\| \|\mathbf{y}\| \sinh \theta$  where  $\theta$  is the hyperbolic angle between  $\mathbf{x}$  and  $\mathbf{y}$ .
- (iv) If  $\mathbf{x}$  and  $\mathbf{y}$  are spacelike vectors satisfying the equality  $|\mathbf{x} \cdot \mathbf{y}| = \|\mathbf{x}\| \|\mathbf{y}\|$ , then  $\mathbf{x} \times \mathbf{y}$  is lightlike.

Now, we define similarity transformation in  $\mathbb{E}_1^3$ . A *pseudo-similarity* (in short *p-similarity*) of Minkowski 3-space  $\mathbb{E}_1^3$  is a decomposition of a homothety (dilatation) a pseudo-orthogonal map and a translation. Let  $\hat{\mathbb{H}}$  be the split quaternion algebra and  $\mathbb{T}\hat{\mathbb{H}}$  be the set of timelike split quaternions such that we identify  $\mathbb{E}_1^3$  with  $\text{Im}\hat{\mathbb{H}}$ .  $\mathbb{T}\hat{\mathbb{H}}$  forms a group under the split quaternion product. A unit timelike split quaternion represents a rotation in the Minkowski 3-space. Therefore, by [17], there exists a unit timelike split quaternion  $q$  such that the transformation  $\mathbf{R}_q : \text{Im}\mathbb{T}\hat{\mathbb{H}} \rightarrow \text{Im}\mathbb{T}\hat{\mathbb{H}}$  defined by

$$\mathbf{R}_q(r) = qrq^{-1}$$

can interpret rotation of a vector in the Minkowski 3-space. Thus, we get

$$f(r) = \mu qrq^{-1} + \mathbf{b} \quad (1)$$

for some fixed  $\mu \neq 0 \in \mathbb{R}$  and  $\mathbf{b} \in \text{Im}\hat{\mathbb{H}} \cong \mathbb{E}_1^3$ . Since  $f$  is a affine map, we get  $\|\vec{f}(\mathbf{u})\| = |\mu| \|\mathbf{u}\|$  for any  $\mathbf{u} \in \mathbb{E}_1^3$  where  $\vec{f}(\overrightarrow{xy}) = \overrightarrow{f(x)f(y)}$  (see [15]). The constant  $|\mu|$  is called a p-similarity ratio of the transformation  $f$ . The p-similarity transformations are a group under the composition of maps and denoted by  $\mathbf{Sim}(\mathbb{E}_1^3)$ . This group is a fundamental group of the Lorentzian similarity geometry. Also, the group of orientation-preserving (reversing) p-similarities are denoted by  $\mathbf{Sim}^+(\mathbb{E}_1^3)$  ( $\mathbf{Sim}^-(\mathbb{E}_1^3)$ , resp. ).

**Theorem 2** The p-similarity transformations preserve the causal characters and angles.

**Proof.** Let  $f$  be a p-similarity. Then, since we can write the equation

$$\vec{f}(\mathbf{u}) \cdot \vec{f}(\mathbf{u}) = \mu^2 (\mathbf{u} \cdot \mathbf{u}), \quad (2)$$

$f$  preserves the causal character in  $\mathbb{E}_1^3$ .

Let  $\mathbf{u}$  and  $\mathbf{v}$  be future-pointing (or past-pointing) timelike vectors and  $\theta, \gamma$  be the angle between  $\mathbf{u}, \mathbf{v}$  and  $\vec{f}(\mathbf{u}), \vec{f}(\mathbf{v})$  respectively. Since  $\vec{f}(\mathbf{u})$  and  $\vec{f}(\mathbf{v})$  have same causal characters with  $\mathbf{u}$  and  $\mathbf{v}$ , we can find the following equation from Theorem 1;

$$\begin{aligned} \vec{f}(\mathbf{u}) \cdot \vec{f}(\mathbf{v}) &= -\|\vec{f}(\mathbf{u})\| \|\vec{f}(\mathbf{v})\| \cosh \gamma \\ \mu^2 (\mathbf{u} \cdot \mathbf{v}) &= -\mu^2 \|\mathbf{u}\| \|\mathbf{v}\| \cosh \gamma \\ -\|\mathbf{u}\| \|\mathbf{v}\| \cosh \theta &= -\|\mathbf{u}\| \|\mathbf{v}\| \cosh \gamma \\ \cosh \theta &= \cosh \gamma. \end{aligned} \quad (3)$$

From here, we have  $\theta = \gamma$ . If  $\mathbf{u}$  and  $\mathbf{v}$  are spacelike vectors satisfying the inequality  $|\mathbf{u} \cdot \mathbf{v}| < \|\mathbf{u}\| \|\mathbf{v}\|$ , then

$$\|\vec{f}(\mathbf{u})\| \|\vec{f}(\mathbf{v})\| = \mu^2 \|\mathbf{u}\| \|\mathbf{v}\| > \mu^2 |\mathbf{u} \cdot \mathbf{v}| = |\vec{f}(\mathbf{u}) \cdot \vec{f}(\mathbf{v})|.$$

Therefore, it can be said from Theorem 1 that we have  $\theta = \gamma$  similar to (3).

It can also be found that  $\theta$  is equal to  $\gamma$  in case of condition (iii) in the Theorem 1. As a consequence, Every p-similarity transformation preserves the angle between any two vectors. ■

### 3 Geometric Invariants of non-lightlike Curves in the Lorentzian Similarity Geometry

Let  $\alpha : t \in I \rightarrow \alpha(t) \in \mathbb{E}_1^3$  be a non-lightlike curve of class  $C^3$  and  $\kappa_\alpha$  and  $\tau_\alpha$  show curvature and torsion of  $\alpha$ , respectively. We denote image of  $\alpha$  under  $f \in \mathbf{Sim}(\mathbb{E}_1^3)$  by  $\beta$ . Then  $\beta$  can be stated as

$$\beta(t) = \mu q \alpha(t) q^{-1} + \mathbf{b} \in \text{Im} \hat{\mathbb{H}}, \quad t \in I. \quad (4)$$

The arc length functions of  $\alpha$  and  $\beta$  starting at  $t_0 \in I$  are

$$s(t) = \int_{t_0}^t \left\| \frac{d\alpha(u)}{du} \right\| du, \quad s^*(t) = \int_{t_0}^t \left\| \frac{d\beta(u)}{du} \right\| du = |\mu| s(t). \quad (5)$$

The Frenet-Serret formulas of  $\alpha$  in the Minkowski 3-space is

$$\frac{d}{ds} \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix} = \begin{bmatrix} 0 & \kappa_\alpha & 0 \\ \varepsilon_{\mathbf{e}_3} \kappa_\alpha & 0 & \tau_\alpha \\ 0 & \varepsilon_{\mathbf{e}_1} \tau_\alpha & 0 \end{bmatrix} \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix} \quad (6)$$

where  $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$  is Frenet frame of  $\alpha$  and  $\varepsilon_{\mathbf{e}_\ell} = \mathbf{e}_\ell \cdot \mathbf{e}_\ell$  for  $1 \leq \ell \leq 3$ . (see [9] and [18]). In this section, the differentiation according to  $s$  is denoted by primes. The curvature  $\kappa_\alpha$  and torsion  $\tau_\alpha$  of non-lightlike curve  $\alpha$  is given by

$$\kappa_\alpha(s) = \|\alpha' \times \alpha''\|, \quad \tau_\alpha(s) = \frac{\det(\alpha', \alpha'', \alpha''')}{\|\alpha' \times \alpha''\|^2}. \quad (7)$$

From (4), (5) and (7), we can calculate the curvature  $\kappa_\beta(|\mu|s)$  and the torsion  $\tau_\beta(|\mu|s)$  as

$$\kappa_\beta = \|\beta' \times \beta''\| = \frac{1}{|\mu|} \kappa_\alpha(s) \quad (8)$$

and

$$\tau_\beta = \frac{1}{\mu} \tau_\alpha(s). \quad (9)$$

Since we have  $ds^* = |\mu| ds$  from (5), we get  $\kappa_\alpha ds = \kappa_\beta ds^*$  and  $|\tau_\alpha| ds = |\tau_\beta| ds^*$ .

Let  $\sigma_\alpha$  and  $\sigma_\beta$  be spherical arc-length parameters of  $\alpha$  and  $\beta$ , respectively. Then, we can find that

$$d\sigma_\alpha = \kappa_\alpha ds = \kappa_\beta ds^* = d\sigma_\beta. \quad (10)$$

Thus, the spherical arc-length element  $d\sigma_\alpha$  is invariant under the group of the p-similarities of  $\mathbb{E}_1^3$ . The derivative formulas of  $\alpha$  with respect to  $\sigma_\alpha$  are given by

$$\frac{d\alpha}{d\sigma_\alpha} = \frac{1}{\kappa_\alpha} \mathbf{e}_1, \quad \frac{d^2\alpha}{d\sigma_\alpha^2} = -\frac{d\kappa_\alpha}{\kappa_\alpha d\sigma_\alpha} \frac{d\alpha}{d\sigma_\alpha} + \frac{1}{\kappa_\alpha} \mathbf{e}_2 \quad (11)$$

and

$$\frac{d}{d\sigma_\alpha} \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \varepsilon_3 & 0 & \frac{\tau_\alpha}{\kappa_\alpha} \\ 0 & \varepsilon_1 \frac{\tau_\alpha}{\kappa_\alpha} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix} \quad (12)$$

by means of (6) and (10). Similarly, for the non-lightlike curve  $\beta$  we also have

$$\frac{d^2\beta}{d\sigma_\beta^2} = -\frac{d\kappa_\beta}{\kappa_\beta d\sigma_\beta} \frac{d\beta}{d\sigma_\beta} + \frac{1}{\kappa_\beta} \mathbf{e}_2^* \quad (13)$$

where  $\{\mathbf{e}_1^*, \mathbf{e}_2^*, \mathbf{e}_3^*\}$  is a Frenet frame field along the non-lightlike curve  $\beta$ . From (8), (9) and (10), we can write

$$-\frac{d\kappa_\beta}{\kappa_\beta d\sigma_\beta} = -\frac{d\kappa_\alpha}{\kappa_\alpha d\sigma_\alpha} \quad \text{and} \quad \frac{\tau_\beta}{\kappa_\beta} = \frac{|\mu|}{\mu} \frac{\tau_\alpha}{\kappa_\alpha}.$$

If we take  $\mu > 0$ , i.e. the p-similarity is an orientation-preserving transformation, we get  $\frac{\tau_\beta}{\kappa_\beta} = \frac{\tau_\alpha}{\kappa_\alpha}$ . Thus, we obtain the following Lemma from above calculations.

**Lemma 3** *The functions  $\tilde{\kappa}_\alpha = -\frac{d\kappa_\alpha}{\kappa_\alpha d\sigma_\alpha}$  and  $\tilde{\tau}_\alpha = \frac{\tau_\alpha}{\kappa_\alpha}$  are invariants under the group of the orientation-preserving p-similarities of the Minkowski 3-space.*

Using (11) and (12) the invariants  $\tilde{\kappa}_\alpha$  and  $\tilde{\tau}_\alpha$  can take the form

$$\tilde{\kappa}_\alpha(\sigma_\alpha) = \frac{\frac{d^2\alpha}{d\sigma_\alpha^2} \cdot \frac{d\alpha}{d\sigma_\alpha}}{\frac{d\alpha}{d\sigma_\alpha} \cdot \frac{d\alpha}{d\sigma_\alpha}}, \quad (14)$$

$$\tilde{\tau}_\alpha(\sigma_\alpha) = \det \left( \frac{d\alpha}{d\sigma_\alpha}, \frac{d^2\alpha}{d\sigma_\alpha^2}, \frac{d^3\alpha}{d\sigma_\alpha^3} \right) \frac{\left\| \frac{d\alpha}{d\sigma_\alpha} \right\|^3}{\left\| \frac{d\alpha}{d\sigma_\alpha} \times \frac{d^2\alpha}{d\sigma_\alpha^2} \right\|^3}. \quad (15)$$

**Definition 4** *Let  $\alpha : I \rightarrow \mathbb{E}_1^3$  be a non-lightlike Frenet curve of the class  $C^3$  parameterized by the spherical arc length parameter  $\sigma_\alpha$ . Let  $\kappa_\alpha(\sigma_\alpha)$  and  $\tau_\alpha(\sigma_\alpha)$  be the curvature and torsion of  $\alpha$ , respectively. The functions*

$$\tilde{\kappa}_\alpha = -\frac{d\kappa_\alpha}{\kappa_\alpha d\sigma_\alpha} \quad \text{and} \quad \tilde{\tau}_\alpha = \frac{\tau_\alpha}{\kappa_\alpha} \quad (16)$$

*are p-shape curvature and p-shape torsion of  $\alpha$ . The ordered pair  $(\tilde{\kappa}_\alpha, \tilde{\tau}_\alpha)$  is called a (local) p-shape of the non-lightlike curve  $\alpha$  in the Minkowski 3-space.*

We consider the pseudo-orthogonal 3-frame  $\{\mathbf{e}_1(\sigma_\alpha)/\kappa_\alpha, \mathbf{e}_2(\sigma_\alpha)/\kappa_\alpha, \mathbf{e}_3(\sigma_\alpha)/\kappa_\alpha\}$ ,  $\sigma_\alpha \in I$ , for the curve. Then, by the equations (11) and (12), we get

$$\frac{d}{d\sigma_\alpha} \begin{bmatrix} \mathbf{e}_1/\kappa_\alpha \\ \mathbf{e}_2/\kappa_\alpha \\ \mathbf{e}_3/\kappa_\alpha \end{bmatrix} = \begin{bmatrix} \tilde{\kappa}_\alpha & 1 & 0 \\ \varepsilon_{\mathbf{e}_3} & \tilde{\kappa}_\alpha & \tilde{\tau}_\alpha \\ 0 & \varepsilon_{\mathbf{e}_1}\tilde{\tau}_\alpha & \tilde{\kappa}_\alpha \end{bmatrix} \begin{bmatrix} \mathbf{e}_1/\kappa_\alpha \\ \mathbf{e}_2/\kappa_\alpha \\ \mathbf{e}_3/\kappa_\alpha \end{bmatrix}. \quad (17)$$

The pseudo-orthogonal frame  $\mathbf{e}_1(\sigma_\alpha)/\kappa_\alpha, \mathbf{e}_2(\sigma_\alpha)/\kappa_\alpha, \mathbf{e}_3(\sigma_\alpha)/\kappa_\alpha$  is invariant under the group  $\mathbf{Sim}^+(\mathbb{E}_1^3)$ . Thus, it may be said that the equation (17) is the Frenet-Serret frame of  $\alpha$  in the Lorentzian similarity 3-space.

### 3.1 The relation between focal curvatures and p-shape of $\alpha$

Let  $\alpha : I \rightarrow \mathbb{E}_1^3$  be a unit speed non-lightlike Frenet curve with the Frenet frame  $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$  and let  $s$  be an arc length parameter of  $\alpha$ . The curve  $\gamma : I \rightarrow \mathbb{E}_1^3$  consisting of the centers of the osculating sphere of the curve  $\alpha$  is called the *focal curve* of  $\alpha$ . The focal curve can be represented by

$$\gamma(s) = \alpha(s) + m_1(s)\mathbf{e}_2 + m_2(s)\mathbf{e}_3$$

where  $m_1$  and  $m_2$  are smooth functions called *focal curvature* of  $\alpha$ . Then, we have the following theorem from [18].

**Theorem 5** Let  $\alpha$  be a non-lightlike curve in  $\mathbb{E}_1^3$ , the radius and center of the osculating sphere of  $\alpha$  at  $\alpha(s)$  are

$$r = \sqrt{(\varepsilon_{\mathbf{e}_2}) \frac{1}{\kappa^2} + (\varepsilon_{\mathbf{e}_3}) \left( \frac{\kappa'}{\kappa^2 \tau} \right)} \quad \text{and} \quad \gamma(s) = \alpha(s) + \frac{\varepsilon_{\mathbf{e}_1} \varepsilon_{\mathbf{e}_2}}{\kappa} \mathbf{e}_2 + \frac{\varepsilon_{\mathbf{e}_1} \varepsilon_{\mathbf{e}_3}}{\tau} \left( \frac{1}{\kappa} \right)' \mathbf{e}_3$$

where  $\mathbf{e}_2$ , and  $\mathbf{e}_3$  are normal and binormal vector fields of the curve at  $\alpha(s)$ .

Using the Theorem 5 we state that the focal curvatures  $m_1$  and  $m_2$  of the non-lightlike curve  $\alpha$  are equal to

$$\frac{\varepsilon_{\mathbf{e}_1} \varepsilon_{\mathbf{e}_2}}{\kappa_\alpha} \quad \text{and} \quad \frac{1}{\tau_\alpha} \left( \frac{\varepsilon_{\mathbf{e}_1} \varepsilon_{\mathbf{e}_3}}{\kappa_\alpha} \right)' \quad (18)$$

respectively. Now, we can show the relation between the focal curvatures and the p-shape curvature and torsion.

**Proposition 6** Let  $\alpha : I \rightarrow \mathbb{E}_1^3$  be a unit speed non-lightlike Frenet curve with the non-zero curvature  $\kappa$  and torsion  $\tau$ . Then,

$$\tilde{\kappa}_\alpha = \varepsilon_{\mathbf{e}_1} \varepsilon_{\mathbf{e}_2} m_1' \quad \text{and} \quad \tilde{\tau}_\alpha = \varepsilon_{\mathbf{e}_1} \varepsilon_{\mathbf{e}_3} \frac{m_1' m_1}{f_2}.$$

**Proof.** From (16) and (18) we can write

$$\tilde{\kappa}_\alpha = -\frac{d\kappa_\alpha}{\kappa_\alpha d\sigma_\alpha} = -\frac{1}{\kappa_\alpha^2} \frac{d\kappa_\alpha}{ds} = \left( \frac{1}{\kappa_\alpha} \right)' = \varepsilon_{\mathbf{e}_1} \varepsilon_{\mathbf{e}_2} m_1'$$

and

$$\tilde{\tau}_\alpha = \frac{\tau_\alpha}{\kappa_\alpha} = \frac{1}{\kappa_\alpha} \tau_\alpha = \varepsilon_{\mathbf{e}_1} \varepsilon_{\mathbf{e}_2} m_1 \frac{\varepsilon_{\mathbf{e}_1} \varepsilon_{\mathbf{e}_3}}{m_2} (\varepsilon_{\mathbf{e}_1} \varepsilon_{\mathbf{e}_2} m_1)' = \varepsilon_{\mathbf{e}_1} \varepsilon_{\mathbf{e}_3} \frac{m_1' m_1}{m_2}.$$

■

## 4 Uniqueness Theorem

Two non-lightlike Frenet curves which have the same torsion and the same positive curvature are always equivalent according to Lorentzian motion. This notion can be extended under the group  $\mathbf{Sim}(\mathbb{E}_1^3)$  for the non-lightlike Frenet curves which have the same p-shape torsion and p-shape curvature, in the Minkowski 3-space  $\mathbb{E}_1^3$ .

**Theorem 7** (Uniqueness Theorem) Let  $\alpha, \alpha^* : I \rightarrow \mathbb{E}_1^3$  be two non-lightlike Frenet curves of class  $C^3$  parameterized by the same spherical arc length parameter  $\sigma$  and have the same causal characters, where  $I \subset \mathbb{R}$  is an open interval. Suppose that  $\alpha$  and  $\alpha^*$  have the same p-shape curvatures  $\tilde{\kappa} = \tilde{\kappa}^*$  and the same p-shape torsions  $\tilde{\tau} = \tilde{\tau}^*$  for any  $\sigma \in I$ .

i) If  $\alpha, \alpha^*$  are timelike curves, there exists a  $f \in \mathbf{Sim}^+(\mathbb{E}_1^3)$  such that  $\alpha^* = f \circ \alpha$ .

ii) If  $\alpha, \alpha^*$  are spacelike curves, there exists a  $f \in \mathbf{Sim}^-(\mathbb{E}_1^3)$  such that  $\alpha^* = f \circ \alpha$ .

**Proof.** Let  $\kappa, \kappa^*$  and  $\tau, \tau^*$  be the curvatures and the torsions of the  $\alpha, \alpha^*$ . Since  $\alpha$  and  $\alpha^*$  have the same shape curvatures  $\tilde{\kappa} = \tilde{\kappa}^*$ , we have

$$\frac{d\kappa}{\kappa} = \frac{d\kappa^*}{\kappa^*} \quad \text{or} \quad \log \kappa = \log \kappa^* + \log \mu$$

where  $\mu \in \mathbb{R}^+$ . Then, we find  $\kappa = \mu\kappa^*$  for any  $\sigma \in I$ . Using  $\tilde{\tau} = \tilde{\tau}^*$  we get  $\tau = \mu\tau^*$  for any  $\sigma \in I$ . Let  $\mathbf{e}_i, \mathbf{e}_i^*, i = 1, 2, 3$ , be a Frenet frame fields on  $\alpha, \alpha^*$  and we choose any point  $\sigma_0 \in I$ . There exists a Lorentzian motion  $\varphi$  of  $\mathbb{E}_1^3$  such that

$$\varphi(\alpha(\sigma_0)) = \alpha^*(\sigma_0) \quad \text{and} \quad \varphi(\mathbf{e}_i(\sigma_0)) = -\varepsilon_{\mathbf{e}_i} \mathbf{e}_i^*(\sigma_0) \quad \text{for } i = 1, 2, 3.$$

Let's consider the function  $\Psi : I \rightarrow \mathbb{R}$  defined by

$$\Psi(\sigma) = \|\varphi(\mathbf{e}_1(\sigma)) + \varepsilon_{\mathbf{e}_1} \mathbf{e}_1^*(\sigma)\|^2 + \|\varphi(\mathbf{e}_2(\sigma)) + \varepsilon_{\mathbf{e}_2} \mathbf{e}_2^*(\sigma)\|^2 + \|\varphi(\mathbf{e}_3(\sigma)) + \varepsilon_{\mathbf{e}_3} \mathbf{e}_3^*(\sigma)\|^2.$$

Then

$$\begin{aligned} \frac{d\Psi}{d\sigma} &= 2 \left( \frac{d}{d\sigma} \varphi(\mathbf{e}_1(\sigma)) + \varepsilon_{\mathbf{e}_1} \frac{d}{d\sigma} \mathbf{e}_1^*(\sigma) \right) \cdot (\varphi(\mathbf{e}_1(\sigma)) + \varepsilon_{\mathbf{e}_1} \mathbf{e}_1^*(\sigma)) \\ &\quad + 2 \left( \frac{d}{d\sigma} \varphi(\mathbf{e}_2(\sigma)) + \varepsilon_{\mathbf{e}_2} \frac{d}{d\sigma} \mathbf{e}_2^*(\sigma) \right) \cdot (\varphi(\mathbf{e}_2(\sigma)) + \varepsilon_{\mathbf{e}_2} \mathbf{e}_2^*(\sigma)) \\ &\quad + 2 \left( \frac{d}{d\sigma} \varphi(\mathbf{e}_3(\sigma)) + \varepsilon_{\mathbf{e}_3} \frac{d}{d\sigma} \mathbf{e}_3^*(\sigma) \right) \cdot (\varphi(\mathbf{e}_3(\sigma)) + \varepsilon_{\mathbf{e}_3} \mathbf{e}_3^*(\sigma)). \end{aligned}$$

Using  $\|\varphi(\mathbf{e}_i)\|^2 = \|\mathbf{e}_i\|^2 = \|\mathbf{e}_i^*\|^2 = 1$  we can write

$$\begin{aligned} \frac{d\Psi}{d\sigma} &= 2\varepsilon_{\mathbf{e}_1} \left[ \left( \varphi \left( \frac{d}{d\sigma} \mathbf{e}_1 \right) \right) \cdot \mathbf{e}_1^* + \varphi(\mathbf{e}_1) \cdot \left( \frac{d}{d\sigma} \mathbf{e}_1^* \right) \right] \\ &\quad + 2\varepsilon_{\mathbf{e}_2} \left[ \left( \varphi \left( \frac{d}{d\sigma} \mathbf{e}_2 \right) \right) \cdot \mathbf{e}_2^* + \varphi(\mathbf{e}_2) \cdot \left( \frac{d}{d\sigma} \mathbf{e}_2^* \right) \right] \\ &\quad + 2\varepsilon_{\mathbf{e}_3} \left[ \left( \varphi \left( \frac{d}{d\sigma} \mathbf{e}_3 \right) \right) \cdot \mathbf{e}_3^* + \varphi(\mathbf{e}_3) \cdot \left( \frac{d}{d\sigma} \mathbf{e}_3^* \right) \right]. \end{aligned}$$

From (12), we get

$$\begin{aligned} \frac{d\Psi}{d\sigma} &= (2\varepsilon_{\mathbf{e}_1} + 2\varepsilon_{\mathbf{e}_2}\varepsilon_{\mathbf{e}_3^*}) [\varphi(\mathbf{e}_2) \cdot \mathbf{e}_1^*] + (2\varepsilon_{\mathbf{e}_1} + 2\varepsilon_{\mathbf{e}_2}\varepsilon_{\mathbf{e}_3}) [\varphi(\mathbf{e}_1) \cdot \mathbf{e}_2^*] \\ &\quad + (2\varepsilon_{\mathbf{e}_2}\tilde{\tau} + 2\varepsilon_{\mathbf{e}_3}\varepsilon_{\mathbf{e}_1^*}\tilde{\tau}^*) [\varphi(\mathbf{e}_3) \cdot \mathbf{e}_2^*] + (2\varepsilon_{\mathbf{e}_2}\tilde{\tau}^* + 2\varepsilon_{\mathbf{e}_3}\varepsilon_{\mathbf{e}_1}\tilde{\tau}) [\varphi(\mathbf{e}_2) \cdot \mathbf{e}_3^*]. \end{aligned}$$

Since  $\alpha$  and  $\alpha^*$  have the same causal characters and  $\tilde{\tau} = \tilde{\tau}^*$ , we can write

$$\begin{aligned} 2\varepsilon_{\mathbf{e}_1} + 2\varepsilon_{\mathbf{e}_2}\varepsilon_{\mathbf{e}_3^*} &= 0, & 2\varepsilon_{\mathbf{e}_1} + 2\varepsilon_{\mathbf{e}_2}\varepsilon_{\mathbf{e}_3} &= 0 \\ 2\varepsilon_{\mathbf{e}_2}\tilde{\tau} + 2\varepsilon_{\mathbf{e}_3}\varepsilon_{\mathbf{e}_1^*}\tilde{\tau}^* &= 0, & 2\varepsilon_{\mathbf{e}_2}\tilde{\tau}^* + 2\varepsilon_{\mathbf{e}_3}\varepsilon_{\mathbf{e}_1}\tilde{\tau} &= 0. \end{aligned}$$

Therefore, we find  $\frac{d\Psi}{d\sigma} = 0$  for any  $\sigma \in I$ . On the other hand, we know  $\Psi(\sigma_0) = 0$  and thus we have  $\Psi(\sigma) = 0$  for any  $\sigma \in I$ . As a result, we can say that

$$\varphi(\mathbf{e}_i(\sigma)) = -\varepsilon_{\mathbf{e}_i} \mathbf{e}_i^*(\sigma), \quad \forall \sigma \in I, \quad i = 1, 2, 3. \quad (19)$$

The map  $g = \mu\varphi : \mathbb{E}_1^3 \rightarrow \mathbb{E}_1^3$  is a p-similarity of  $\mathbb{E}_1^3$ . We examine an other function  $\Phi : I \rightarrow \mathbb{R}$  such that

$$\Phi(\sigma) = \left\| \frac{d}{d\sigma} g(\alpha(\sigma)) + \varepsilon_{\mathbf{e}_1} \frac{d}{d\sigma} \alpha^*(\sigma) \right\|^2 \quad \text{for } \forall \sigma \in I.$$

Taking derivative of this function with respect to  $\sigma$  we get

$$\begin{aligned} \frac{d\Phi}{d\sigma} &= 2g \left( \frac{d^2\alpha}{d\sigma^2} \right) \cdot g \left( \frac{d\alpha}{d\sigma} \right) + 2\varepsilon_{\mathbf{e}_1} \left[ g \left( \frac{d^2\alpha}{d\sigma^2} \right) \cdot \frac{d\alpha^*}{d\sigma} \right] \\ &\quad + 2\varepsilon_{\mathbf{e}_1} \frac{d^2\alpha^*}{d\sigma^2} \cdot g \left( \frac{d\alpha}{d\sigma} \right) + 2 \left[ \frac{d^2\alpha^*}{d\sigma^2} \cdot \frac{d\alpha^*}{d\sigma} \right]. \end{aligned}$$

Since the function  $\varphi$  is linear map and we have (11) and (19), we can write

$$\frac{d\Phi}{d\sigma} = 2\varepsilon_{\mathbf{e}_1^*} \mu^2 \frac{\tilde{\kappa}}{\kappa^2} - 2\varepsilon_{\mathbf{e}_1^*} \mu \frac{\tilde{\kappa}}{\kappa\kappa^*} - 2\varepsilon_{\mathbf{e}_1^*} \mu \frac{\tilde{\kappa}^*}{\kappa\kappa^*} + 2\varepsilon_{\mathbf{e}_1^*} \frac{\tilde{\kappa}^*}{(\kappa^*)^2}.$$

Using  $\mu = \frac{\kappa}{\kappa^*}$ , we have  $\frac{d\Phi}{d\sigma} = 0$ . Also, we can find

$$\frac{d}{d\sigma} g(\alpha(\sigma_0)) = g\left(\frac{1}{\kappa} \mathbf{e}_1(\sigma_0)\right) = -\varepsilon_{\mathbf{e}_1} \frac{1}{\kappa^*} \mathbf{e}_1^*(\sigma_0)$$

and we know

$$\frac{d}{d\sigma} \alpha^*(\sigma_0) = \frac{1}{\kappa^*} \mathbf{e}_1^*(\sigma_0).$$

Then, we conclude that  $\Phi(\sigma_0) = 0$ . Hence,  $\Phi(\sigma) = 0$  for  $\forall \sigma \in I$ . This means that

$$\frac{d}{d\sigma} g(\alpha(\sigma)) = -\varepsilon_{\mathbf{e}_1} \frac{d}{d\sigma} \alpha^*(\sigma)$$

or equivalently  $\alpha^*(\sigma) = -\varepsilon_{\mathbf{e}_1} g(\alpha(\sigma)) + \mathbf{b}$  where  $\mathbf{b}$  is a constant vector. Then, the image of non-lightlike curve  $\alpha$  under the p-similarity  $f = \vartheta \circ (-\varepsilon_{\mathbf{e}_1} g)$ , where  $\vartheta : \mathbb{E}_1^3 \rightarrow \mathbb{E}_1^3$  is a translation function determined by  $\mathbf{b}$ , is the non-lightlike curve  $\alpha^*$ . If the curves  $\alpha, \alpha^*$  are taken as the timelike curves, the p-similarity transformation  $f$  is an orientation-preserving transformation. Also, when the curves  $\alpha, \alpha^*$  are the spacelike curves, the p-similarity transformation  $f$  is an orientation-reversing transformation. ■

Is it possible to say that two spacelike Frenet curves are equivalent under orientation-preserving p-similarity? We can see the answer with the following theorem.

**Theorem 8** *Let  $\alpha, \alpha^* : I \rightarrow \mathbb{E}_1^3$  be two spacelike Frenet curves of class  $C^3$  parameterized by the same spherical arc length parameter  $\sigma$ , where  $I \subset \mathbb{R}$  is an open interval. Suppose that  $\alpha$  and  $\alpha^*$  have the same p-shape curvature  $\tilde{\kappa} = \tilde{\kappa}^*$  and  $\tilde{\tau} = -\tilde{\tau}^*$  for the p-shape torsions  $\tilde{\tau}, \tilde{\tau}^*$ . Then there exists an orientation-preserving p-similarity  $f$  of  $\mathbb{E}_1^3$  such that  $\alpha^* = f \circ \alpha$ .*

**Proof.** The proof is similar to the proof of the Theorem 7. Let  $\mathbf{e}_i, \mathbf{e}_i^*, i = 1, 2, 3$ , be a Frenet frame field on  $\alpha, \alpha^*$  and we choose any point  $\sigma_0 \in I$ . If  $\mathbf{e}_2$  and  $\mathbf{e}_2^*$  are timelike vectors, There exists a Lorentzian motion  $\varphi$  of  $\mathbb{E}_1^3$  such that

$$\varphi(\alpha(\sigma_0)) = \alpha^*(\sigma_0), \quad \varphi(\mathbf{e}_1(\sigma_0)) = \mathbf{e}_1^*(\sigma_0) \quad \text{and} \quad \varphi(\mathbf{e}_i(\sigma_0)) = -\mathbf{e}_i^*(\sigma_0) \quad \text{for } i = 2, 3.$$

Let's consider the function  $\Psi : I \rightarrow \mathbb{R}$  defined by

$$\Psi(\sigma) = \|\varphi(\mathbf{e}_1(\sigma)) - \mathbf{e}_1^*(\sigma)\|^2 + \|\varphi(\mathbf{e}_2(\sigma)) + \mathbf{e}_2^*(\sigma)\|^2 + \|\varphi(\mathbf{e}_3(\sigma)) + \mathbf{e}_3^*(\sigma)\|^2.$$

Then

$$\frac{d\Psi}{d\sigma} = 2(\tilde{\tau} + \tilde{\tau}^*)(\varphi(\mathbf{e}_3) \cdot \mathbf{e}_2 + \varphi(\mathbf{e}_2) \cdot \mathbf{e}_3) = 0.$$

Due to  $\Psi(\sigma_0) = 0$ , we can write

$$\varphi(\mathbf{e}_1(\sigma)) = \mathbf{e}_1^*(\sigma) \quad \text{and} \quad \varphi(\mathbf{e}_i(\sigma)) = -\mathbf{e}_i^*(\sigma) \quad \text{for } i = 2, 3 \quad \forall \sigma \in I.$$

The map  $g = \mu\varphi : \mathbb{E}_1^3 \rightarrow \mathbb{E}_1^3$  is a p-similarity of  $\mathbb{E}_1^3$ . We examine the function  $\Phi : I \rightarrow \mathbb{R}$  such that

$$\Phi(\sigma) = \left\| \frac{d}{d\sigma} g(\alpha(\sigma)) - \frac{d}{d\sigma} \alpha^*(\sigma) \right\|^2 \quad \text{for } \forall \sigma \in I.$$



Since we have  $\frac{d\Phi}{d\sigma} = 0$  and  $\Phi(\sigma_0) = 0$ , we get  $\Phi(\sigma) = 0$  for any  $\sigma \in I$ . Namely, we can write  $\frac{d}{d\sigma}g(\alpha(\sigma)) = \frac{d}{d\sigma}\alpha^*(\sigma)$  or equivalently  $\alpha^*(\sigma) = g(\alpha(\sigma)) + \mathbf{b}$  where  $\mathbf{b}$  is a constant vector. So, we have  $f = \vartheta \circ g$ , where  $\vartheta : \mathbb{E}_1^3 \rightarrow \mathbb{E}_1^3$  is a translation function determined by  $\mathbf{b}$ , is orientation-preserving p-similarity transformation such that the image of the spacelike curve  $\alpha$  under  $f$  is the spacelike curve  $\alpha^*$ , i.e.  $\alpha^* = f \circ \alpha$ .

In the same way, if we take  $\mathbf{e}_3$  and  $\mathbf{e}_3^*$  as timelike vectors, we can find an orientation-preserving p-similarity  $f$  which provides  $\alpha^* = f \circ \alpha$  such that the functions  $\Psi$  and  $\Phi$  are respectively defined by

$$\begin{aligned}\Psi(\sigma) &= \|\varphi(\mathbf{e}_1(\sigma)) - \mathbf{e}_1^*(\sigma)\|^2 + \|\varphi(\mathbf{e}_2(\sigma)) - \mathbf{e}_2^*(\sigma)\|^2 + \|\varphi(\mathbf{e}_3(\sigma)) - \mathbf{e}_3^*(\sigma)\|^2, \\ \Phi(\sigma) &= \left\| \frac{d}{d\sigma}g(\alpha(\sigma)) - \frac{d}{d\sigma}\alpha^*(\sigma) \right\|^2 \quad \text{for } \forall \sigma \in I.\end{aligned}$$

■

## 5 Construction of the non-lightlike Frenet curves by curves on the Lorentzian and hyperbolic unit sphere

Let  $\mathbf{c} : I \rightarrow S_1^2$  be non-lightlike spherical curve with the arc length parameter  $\sigma$ . The orthonormal frame  $\{\mathbf{c}(\sigma), \mathbf{t}(\sigma), \mathbf{q}(\sigma)\}$  along  $\mathbf{c}$  is called the *Sabban frame* of  $\mathbf{c}$  if  $\mathbf{t}(\sigma) = \frac{d\mathbf{c}}{d\sigma}$  is the unit tangent vector of  $\mathbf{c}$  and  $\mathbf{q}(\sigma) = \mathbf{c}(\sigma) \times \mathbf{t}(\sigma)$ . Then we state spherical Frenet-Serret formulas of the non-lightlike curve  $\mathbf{c}$ .

If the curve  $\mathbf{c}$  is a timelike curve, i.e.  $\mathbf{t}(\sigma)$  is timelike vector, we have the following spherical Frenet-Serret formulas of  $\mathbf{c}$ :

$$\frac{d}{d\sigma} \begin{bmatrix} \mathbf{c} \\ \mathbf{t} \\ \mathbf{q} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & k_g \\ 0 & k_g & 0 \end{bmatrix} \begin{bmatrix} \mathbf{c} \\ \mathbf{t} \\ \mathbf{q} \end{bmatrix} \quad (20)$$

If  $\mathbf{q}(\sigma)$  is a timelike vector, we have the following spherical Frenet-Serret formulas of  $\mathbf{c}$ :

$$\frac{d}{d\sigma} \begin{bmatrix} \mathbf{c} \\ \mathbf{t} \\ \mathbf{q} \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & k_g \\ 0 & k_g & 0 \end{bmatrix} \begin{bmatrix} \mathbf{c} \\ \mathbf{t} \\ \mathbf{q} \end{bmatrix} \quad (21)$$

If  $\mathbf{c} : I \rightarrow H_0^2$  is a spacelike spherical curve with the arc length parameter  $\sigma$ , then spherical Frenet-Serret formulas of  $\mathbf{c}$  are

$$\frac{d}{d\sigma} \begin{bmatrix} \mathbf{c} \\ \mathbf{t} \\ \mathbf{q} \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & k_g \\ 0 & -k_g & 0 \end{bmatrix} \begin{bmatrix} \mathbf{c} \\ \mathbf{t} \\ \mathbf{q} \end{bmatrix} \quad (22)$$

since  $\mathbf{c}(\sigma)$  is a timelike vector.  $k_g(\sigma) = \varepsilon_{\mathbf{q}} \det \left( \mathbf{c}(\sigma), \mathbf{t}(\sigma), \frac{d\mathbf{t}}{d\sigma}(\sigma) \right)$  is the geodesic curvature of  $\mathbf{c}$  for three different spherical Frenet-Serret formulas.

Let  $k : I \rightarrow \mathbb{R}$  be a function of class  $C^1$ . We can describe a non-lightlike curve  $\alpha : I \rightarrow \mathbb{E}_1^3$  given by

$$\alpha(\sigma) = b \int e^{\int k(\sigma) d\sigma} \mathbf{c}(\sigma) d\sigma + \mathbf{a}, \quad (23)$$

where  $\mathbf{a}$  is a constant vector and  $b$  is a real constant. The fact that  $\sigma$  is arc spherical length parameter of  $\alpha$  can be easily seen because we have  $\frac{\frac{d\alpha}{d\sigma}}{\left\| \frac{d\alpha}{d\sigma} \right\|} = \mathbf{c}(\sigma)$ . Then, we can state a description of all Frenet curves in Minkowski 3-space.

**Proposition 9** *The non-lightlike curve  $\alpha$  defined by (23) is a Frenet curve with shape curvature  $\tilde{\kappa} = k(\sigma)$  and shape torsion  $\tilde{\tau} = \varepsilon_{\mathbf{q}} k_g(\sigma)$  in the Minkowski 3-space. Furthermore, all non-lightlike Frenet curves can be obtained in this way.*

**Proof.** First, from (23) we can write

$$\begin{aligned}\frac{d\alpha}{d\sigma} &= b e^{\int k(\sigma) d\sigma} \mathbf{c}(\sigma), \quad \frac{d^2\alpha}{d\sigma^2} = b e^{\int k(\sigma) d\sigma} \left[ k(\sigma) \mathbf{c}(\sigma) + \frac{d\mathbf{c}}{d\sigma} \right] \\ \frac{d^3\alpha}{d\sigma^3} &= b e^{\int k(\sigma) d\sigma} \left[ \left\{ k^2(\sigma) + \frac{dk}{d\sigma} \right\} \mathbf{c}(\sigma) + 2k(\sigma) \frac{d\mathbf{c}}{d\sigma} + \frac{d^2\mathbf{c}}{d\sigma^2} \right].\end{aligned}$$

Then, because of the equation

$$\frac{d\alpha}{d\sigma} \times \frac{d^2\alpha}{d\sigma^2} = b^2 e^{2\int k(\sigma) d\sigma} \left( \mathbf{c}(\sigma) \times \frac{d\mathbf{c}}{d\sigma} \right) \neq 0,$$

we have  $\alpha$  is non-lightlike Frenet curve. Using (14) and (15) we find that

$$\tilde{\kappa} = k(\sigma) \quad \text{and} \quad \tilde{\tau} = \det \left( \mathbf{c}, \frac{d\mathbf{c}}{d\sigma}, \frac{d\mathbf{t}}{d\sigma} \right) = \varepsilon_{\mathbf{q}} k_g(\sigma).$$

Conversely, suppose that  $\alpha : I \rightarrow \mathbb{E}_1^3$  is a non-lightlike regular curve parameterized by a spherical arc length parameter  $\sigma$ . Denote by  $\kappa(\sigma)$  and  $\tau(\sigma)$  the curvature and the torsion of  $\mathbf{c}$ , respectively. Let  $\mathbf{c}$  be the spherical indicator of  $\alpha$  such that  $\mathbf{c} : I \rightarrow \mathbb{E}_1^3$  is given by

$$\mathbf{c}(\sigma) = \mathbf{e}_1(\sigma) = \frac{\frac{d\alpha}{d\sigma}}{\left\| \frac{d\alpha}{d\sigma} \right\|} = \kappa(\sigma) \frac{d\alpha}{d\sigma}. \quad (24)$$

We can say that  $\sigma$  is an arc length parameter of  $\mathbf{c}$  and  $k_g = \varepsilon_{\mathbf{q}} \det \left( \mathbf{c}(\sigma), \mathbf{t}(\sigma), \frac{d\mathbf{t}(\sigma)}{d\sigma} \right) = \varepsilon_{\mathbf{q}} \tilde{\tau}$  is the geodesic curvature of  $\mathbf{c}$ . If we take  $k(\sigma) = \tilde{\kappa}(\sigma)$ , then

$$\begin{aligned}\int e^{\int k(\sigma) d\sigma} \mathbf{c}(\sigma) d\sigma &= \int e^{\int -\frac{d\kappa}{\kappa d\sigma} d\sigma} \mathbf{c}(\sigma) d\sigma = e^{b_0} \int \frac{1}{\kappa} \mathbf{c}(\sigma) d\sigma \\ &= e^{b_0} \int \frac{d\alpha}{d\sigma} d\sigma = e^{b_0} \alpha(\sigma) + \mathbf{a}_0\end{aligned}$$

where  $b_0$  is a real constant and  $\mathbf{a}_0$  is a constant vector. Hence, we can write

$$\alpha(\sigma) = b \int e^{\int k(\sigma) d\sigma} \mathbf{c}(\sigma) d\sigma + \mathbf{a}.$$

■

**Theorem 10** (Existence Theorem) *Let  $z_i : I \rightarrow \mathbb{R}$ ,  $i = 1, 2$ , be two functions of class  $C^1$  and  $\mathbf{e}_1^0, \mathbf{e}_2^0, \mathbf{e}_3^0$  be an right-handed orthonormal triad of vectors at a point  $x_0$  in the Minkowski 3-space  $\mathbb{E}_1^3$ . According to a  $p$ -similarity with center  $x_0$  there exists a unique non-lightlike curve  $\alpha : I \rightarrow \mathbb{E}_1^3$  such that  $\alpha$  satisfies the following conditions:*

- (i) *There exists a  $\sigma_0 \in I$  such that  $\alpha(\sigma_0) = x_0$  and the Frenet-Serret frame of  $\alpha$  at  $x_0$  is  $\{\mathbf{e}_1^0, \mathbf{e}_2^0, \mathbf{e}_3^0\}$ .*
- (ii)  *$\tilde{\kappa}(\sigma) = z_1(\sigma)$  and  $\tilde{\tau}(\sigma) = \varepsilon_{\mathbf{e}_3^0} z_2(\sigma)$  for any  $\sigma \in I$ .*

**Proof.** We consider the system of differential equations

$$\frac{d\mathbf{X}}{d\sigma}(\sigma) = \mathbf{M}(\sigma) \mathbf{X}(\sigma) \quad (25)$$

where  $\mathbf{X}(\sigma) = [\mathbf{c}(\sigma) \ \mathbf{t}(\sigma) \ \mathbf{q}(\sigma)]$  and  $\mathbf{M}$  is one of the following matrices depending on whether  $\mathbf{t}(\sigma)$ ,  $\mathbf{c}(\sigma)$  or  $\mathbf{q}(\sigma)$  is a timelike vector, respectively:

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & z_2 \\ 0 & z_2 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & z_2 \\ 0 & -z_2 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & z_2 \\ 0 & z_2 & 0 \end{bmatrix}.$$

The system (25) has an unique solution  $\mathbf{X}(\sigma)$  which satisfies initial conditions  $\mathbf{X}(\sigma_0) = [\mathbf{e}_1^0 \ \mathbf{e}_2^0 \ \mathbf{e}_3^0]$  for  $\sigma_0 \in I$ . If  $\mathbf{I}$  is the unit matrix and  $\mathbf{X}^t$  is the transposed matrix of  $\mathbf{X}(\sigma)$ , then we can obtain

$$\begin{aligned} \frac{d}{d\sigma} (\mathbf{I}^* \mathbf{X}^t \mathbf{I}^* \mathbf{X}) &= \mathbf{I}^* \frac{d}{d\sigma} \mathbf{X}^t \mathbf{I}^* \mathbf{X} + \mathbf{I}^* \mathbf{X}^t \mathbf{I}^* \frac{d}{d\sigma} \mathbf{X} \\ &= \mathbf{I}^* \mathbf{X}^t \mathbf{M}^t \mathbf{I}^* \mathbf{X} + \mathbf{I}^* \mathbf{X}^t \mathbf{I}^* \mathbf{M} \mathbf{X} \\ &= \mathbf{I}^* \mathbf{X}^t (\mathbf{M}^t \mathbf{I}^* + \mathbf{I}^* \mathbf{M}) \mathbf{X} = 0 \end{aligned}$$

using the equation  $\mathbf{M}^t \mathbf{I}^* + \mathbf{I}^* \mathbf{M} = [0]_{3 \times 3}$ , where  $\mathbf{I}^* = \text{diag}(-1, 1, 1)$ ,  $\text{diag}(1, -1, 1)$  or  $\text{diag}(1, 1, -1)$ , when  $\mathbf{c}(\sigma)$ ,  $\mathbf{t}(\sigma)$  or  $\mathbf{q}(\sigma)$  is a timelike vector, respectively. Also, we have  $\mathbf{I}^* \mathbf{X}^t(\sigma_0) \mathbf{I}^* \mathbf{X}(\sigma_0) = \mathbf{I}$  since  $\{\mathbf{e}_1^0, \mathbf{e}_2^0, \mathbf{e}_3^0\}$  is the orthonormal frame. As a result, we find  $\mathbf{I}^* \mathbf{X}^t(\sigma) \mathbf{I}^* \mathbf{X}(\sigma) = \mathbf{I}$  for any  $\sigma \in I$ . This means that the vector fields  $\mathbf{t}(\sigma)$ ,  $\mathbf{c}(\sigma)$  and  $\mathbf{q}(\sigma)$  form a right-handed orthonormal frame field.

Let  $\alpha : I \rightarrow \mathbb{E}_1^3$  be the regular non-lightlike curve given by

$$\alpha(\sigma) = b \int_{\sigma_0}^{\sigma} e^{\int z_1(\sigma) d\sigma} \mathbf{c}(\sigma) d\sigma + x_0, \quad \sigma \in I, \ b > 0.$$

By the proposition (9), we get that the Frenet-Serret frame field of  $\alpha$  is

$$\{\mathbf{e}_1(\sigma) = \mathbf{c}(\sigma), \ \mathbf{e}_2(\sigma) = \mathbf{t}(\sigma), \ \mathbf{e}_3(\sigma) = \mathbf{q}(\sigma)\}$$

and Frenet-Serret frame of  $\alpha$  at  $x_0 = \alpha(\sigma_0)$  is

$$\{\mathbf{e}_1^0(\sigma_0) = \mathbf{c}(\sigma_0), \ \mathbf{e}_2^0(\sigma_0) = \mathbf{t}(\sigma_0), \ \mathbf{e}_3^0(\sigma_0) = \mathbf{q}(\sigma_0)\}.$$

Besides, the functions  $z_1$  and  $\varepsilon_{\mathbf{e}_3^0} z_2$  are the p-shape curvature and p-shape torsion of  $\alpha$ , respectively. ■

From Theorems 7 and 10, we get the following theorem which is an analogue of the fundamental theorem of curves.

**Theorem 11** *Let  $z_i : I \rightarrow \mathbb{R}$ ,  $i = 1, 2$ , be two functions of class  $C^1$ . According to p-similarity there exists a unique non-lightlike Frenet curve with p-shape curvature  $z_1$  and p-shape torsion  $z_2$ .*

### 5.1 Forming a non-lightlike curve from its p-shape

Let  $\alpha : I \rightarrow \mathbb{E}_1^3$  be a non-lightlike curve with the spherical arc length parameter  $\sigma$  such that the ordered pair  $(\tilde{\kappa}_\alpha, \tilde{\tau}_\alpha)$  is p-shape of the  $\alpha$  defined by (16). From the Theorem 11 we have that  $\alpha$  is uniquely determined by its p-shape according to p-similarity in the Minkowski 3-space. First we define fixed right-handed orthonormal triad of non-lightlike vectors  $\mathbf{e}_1^0, \mathbf{e}_2^0, \mathbf{e}_3^0$ . When  $\mathbf{t}(\sigma)$ ,  $\mathbf{c}(\sigma)$  or  $\mathbf{q}(\sigma)$  is timelike vector, we take respectively differential equations

$$\frac{d\mathbf{c}}{d\sigma} = \mathbf{t}(\sigma), \quad \frac{d\mathbf{t}}{d\sigma} = \mathbf{c}(\sigma) + \tilde{\tau}_\alpha \mathbf{q}(\sigma), \quad \frac{d\mathbf{q}}{d\sigma} = \tilde{\tau}_\alpha \mathbf{t}(\sigma) \quad (26)$$

$$\frac{d\mathbf{c}}{d\sigma} = -\mathbf{t}(\sigma), \quad \frac{d\mathbf{t}}{d\sigma} = -\mathbf{c}(\sigma) + \tilde{\tau}_\alpha \mathbf{q}(\sigma), \quad \frac{d\mathbf{q}}{d\sigma} = -\tilde{\tau}_\alpha \mathbf{t}(\sigma) \quad (27)$$

$$\frac{d\mathbf{c}}{d\sigma} = -\mathbf{t}(\sigma), \quad \frac{d\mathbf{t}}{d\sigma} = \mathbf{c}(\sigma) - \tilde{\tau}_\alpha \mathbf{q}(\sigma), \quad \frac{d\mathbf{q}}{d\sigma} = -\tilde{\tau}_\alpha \mathbf{t}(\sigma). \quad (28)$$

The unique solution of one of these differential equations with initial conditions  $\mathbf{e}_1^0, \mathbf{e}_2^0, \mathbf{e}_3^0$ , determine a spherical non-lightlike curve  $\mathbf{c} = \mathbf{c}(\sigma)$  such that  $\mathbf{c}(\sigma_0) = \mathbf{e}_1^0$  for some  $\sigma_0 \in I$ . Let  $\rho(\sigma) = \int_{\sigma_1}^{\sigma} \tilde{\kappa}_{\alpha}(\sigma) d\sigma$  for fixed  $\sigma_1 \in I$ . Using the equation (23) and proposition 9 we can find the non-lightlike curve

$$\alpha(\sigma) = \alpha_0 + \int_{\sigma_0}^{\sigma} e^{\rho(\sigma)} \mathbf{c}(\sigma) d\sigma \quad (29)$$

passes through a point  $\alpha_0 = \alpha(\sigma_0)$ . Now, we show a few examples of the non-lightlike curves constructed by above procedure.

**Example 12** Let  $p$ -shape  $(\tilde{\kappa}_{\alpha}, \tilde{\tau}_{\alpha})$  of the  $\alpha : I \rightarrow \mathbb{E}_1^3$  be  $(0, a)$ , where  $a \neq 0$  is real constant. We can find  $\rho(\sigma) = 0$  for any  $\sigma \in I$ .

i) We take the unit vector  $\mathbf{t}(\sigma)$  as timelike vector. Choose initial conditions

$$\mathbf{e}_1^0 = \left(0, -\frac{1}{\sqrt{1+a^2}}, \frac{a}{\sqrt{1+a^2}}\right), \mathbf{e}_2^0 = (1, 0, 0), \mathbf{e}_3^0 = \left(0, \frac{a}{\sqrt{1+a^2}}, \frac{1}{\sqrt{1+a^2}}\right). \quad (30)$$

Then, the system (26) describes a spherical timelike curve  $\mathbf{c} : I \rightarrow S_1^2$  defined by

$$\mathbf{c}(\sigma) = \left(\frac{1}{\sqrt{1+a^2}} \sinh(\sqrt{1+a^2}\sigma), -\frac{1}{\sqrt{1+a^2}} \cosh(\sqrt{1+a^2}\sigma), \frac{a}{\sqrt{1+a^2}}\right) \quad (31)$$

with  $\mathbf{c}(0) = \mathbf{e}_1^0$ , in the Minkowski 3-space. Solving the equation (29) we obtain the spacelike curve parameterized by

$$\alpha(\sigma) = \left(\frac{1}{1+a^2} \cosh(\sqrt{1+a^2}\sigma), -\frac{1}{1+a^2} \sinh(\sqrt{1+a^2}\sigma), \frac{a}{\sqrt{1+a^2}}\sigma\right), \quad \sigma \in I.$$

ii) Let the unit vector  $\mathbf{c}(\sigma)$  be timelike vector. We choose another initial conditions

$$\mathbf{e}_1^0 = \left(\frac{a}{\sqrt{a^2-1}}, 0, \frac{1}{\sqrt{a^2-1}}\right), \mathbf{e}_2^0 = (0, 1, 0), \mathbf{e}_3^0 = \left(\frac{1}{\sqrt{a^2-1}}, 0, \frac{a}{\sqrt{a^2-1}}\right)$$

where  $a^2 > 1$ . Then, the system (27) describes a spherical spacelike curve  $\mathbf{c} : I \rightarrow H_0^2$  defined by

$$\mathbf{c}(\sigma) = \left(\frac{a}{\sqrt{a^2-1}}, \frac{1}{\sqrt{a^2-1}} \sin(\sqrt{a^2-1}\sigma), \frac{1}{\sqrt{a^2-1}} \cos(\sqrt{a^2-1}\sigma)\right) \quad (32)$$

with  $\mathbf{c}(0) = \mathbf{e}_1^0$ , in the Minkowski 3-space. Solving the equation (29) we obtain the timelike curve given by

$$\alpha(\sigma) = \left(\frac{a}{\sqrt{a^2-1}}\sigma, -\frac{1}{a^2-1} \cos(\sqrt{a^2-1}\sigma), \frac{1}{a^2-1} \sin(\sqrt{a^2-1}\sigma)\right).$$

iii) Let the unit vector  $\mathbf{q}(\sigma)$  be timelike vector. Choose another initial conditions

$$\mathbf{e}_1^0 = \left(\frac{1}{\sqrt{a^2-1}}, 0, \frac{a}{\sqrt{a^2-1}}\right), \mathbf{e}_2^0 = (0, 1, 0), \mathbf{e}_3^0 = \left(\frac{a}{\sqrt{a^2-1}}, 0, \frac{1}{\sqrt{a^2-1}}\right)$$

where  $a^2 > 1$ . Then, the system (28) describes a spherical spacelike curve  $\mathbf{c} : I \rightarrow S_1^2$  defined by

$$\mathbf{c}(\sigma) = \left(\frac{1}{\sqrt{a^2-1}} \cosh(\sqrt{a^2-1}\sigma), \frac{1}{\sqrt{a^2-1}} \sinh(\sqrt{a^2-1}\sigma), \frac{a}{\sqrt{a^2-1}}\right) \quad (33)$$

with  $\mathbf{c}(0) = \mathbf{e}_1^0$ , in the Minkowski 3-space. Solving the equation (29) we obtain the spacelike Frenet curve given by

$$\alpha(\sigma) = \left(\frac{1}{a^2-1} \sinh(\sqrt{a^2-1}\sigma), \frac{1}{a^2-1} \cosh(\sqrt{a^2-1}\sigma), \frac{a}{\sqrt{a^2-1}}\sigma\right).$$

**Example 13** Let  $\alpha : I \rightarrow \mathbb{E}_1^3$  be a non-lightlike curve with p-shape  $(\tilde{\kappa}_\alpha, \tilde{\tau}_\alpha) = (1/\sigma, a)$  where  $a \neq 0$  is real constants. Because of  $\rho(\sigma) = \ln \sigma$ , the parametric equation of the non-lightlike curve  $\alpha$  is given by

$$\alpha(\sigma) = \left( \frac{t \cosh t - \sinh t}{(1+a^2)^{3/2}}, \frac{\cosh t - t \sinh t}{(1+a^2)^{3/2}}, \frac{at^2}{2(1+a^2)^{3/2}} \right)$$

where  $t = \sqrt{1+a^2}\sigma$ . As in the Example 12 we take the same spherical timelike curve  $\mathbf{c} = \mathbf{c}(\sigma)$  parameterized by (31).

Now, we study non-lightlike self-similar curves in  $\mathbb{E}_1^3$ . A non-lightlike curve  $\alpha : I \rightarrow \mathbb{E}_1^3$  is called *self-similar* if any p-similarity  $f \in G$  conserve globally  $\alpha$  and  $G$  acts transitively on  $\alpha$  where  $G$  is a one-parameter subgroup of  $\mathbf{Sim}(\mathbb{E}_1^3)$ . This means that p-shape curvatures are constant. In fact, let  $p_1 = \alpha(s_1)$  and  $p_2 = \alpha(s_2)$  be two different points lying on  $\alpha$ . Since  $G$  acts transitively on  $\alpha$ , there is a similarity  $f \in G$  such that  $f(p_1) = p_2$ . Then, we find  $\tilde{\kappa}_\alpha(s_1) = \tilde{\kappa}_\alpha(s_2)$  and  $\tilde{\tau}_\alpha(s_1) = \tilde{\tau}_\alpha(s_2)$  because of the invariance of p-shape curvatures.

Now, we shall state the parametrization of all non-lightlike self-similar curves. Let  $\alpha : I \rightarrow \mathbb{E}_1^3$  be a non-lightlike curve with the p-shape  $(\tilde{\kappa}_\alpha, \tilde{\tau}_\alpha) = (b, a)$  where  $a \neq 0$  and  $b \neq 0$  are real constants. Firstly we take  $\mathbf{t}(\sigma)$  as a timelike unit vector. Choosing initial conditions (30) as in the Example 12, we get the same spherical timelike curve (31) which is a pseudo-circle with a radius  $1/\sqrt{1+a^2}$ . Also, we have  $\rho(\sigma) = \int_0^\sigma b d\sigma = b\sigma$  for  $\sigma \in I$ . Solving the equation (29), we obtain a spacelike self-similar curve which has the spherical arc length parametrization as the following

$$\alpha_{\mathbf{t}}(\sigma) = \left( \frac{e^{b\sigma}}{(b^2 - q^2)} \left( \frac{b}{q} \sinh(q\sigma) - \cosh(q\sigma) \right), \frac{e^{b\sigma}}{(b^2 - q^2)} \left( \sinh(q\sigma) - \frac{b}{q} \cosh(q\sigma) \right), \frac{a}{bq} e^{b\sigma} \right)$$

where  $q = \sqrt{1+a^2}$ .

If we take  $\mathbf{c}(\sigma)$  as a timelike unit vector, by using (32) we obtain similarly a timelike self-similar curve given by

$$\alpha_{\mathbf{c}}(\sigma) = \left( \frac{a}{bn} e^{b\sigma}, \frac{e^{b\sigma}}{(b^2 - n^2)} \left( \frac{b}{n} \sin(n\sigma) - \cos(n\sigma) \right), \frac{e^{b\sigma}}{(b^2 - n^2)} \left( \frac{b}{n} \cos(n\sigma) + \sin(n\sigma) \right) \right)$$

where  $n = \sqrt{a^2 - 1}$ .

If we take  $\mathbf{q}(\sigma)$  as a timelike unit vector, by using (33) we get a spacelike self-similar curve as the following

$$\alpha_{\mathbf{q}}(\sigma) = \left( \frac{e^{b\sigma}}{(b^2 - n^2)} \left( \frac{b}{n} \cosh(n\sigma) - \sinh(n\sigma) \right), \frac{e^{b\sigma}}{(b^2 - n^2)} \left( \frac{b}{n} \sinh(n\sigma) - \cosh(n\sigma) \right), \frac{a}{bn} e^{b\sigma} \right).$$

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